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Best L^1 Approximation by Weak Chebyshev Systems and the Uniqueness of Interpolating Perfect Splines

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INTRODUCTION

The following theorem of Hobby and Rice [6] plays a central role in this paper.

THEOREM (Hobby–Rice). *Let $\{u_1(t),..., u_n(t)\}$ be any sequence of linearly independent functions in $L^1[0, 1]$. Then there exist points $0 = \tau_0 < \tau_1 < \cdots < \tau_k < \tau_{k+1} = 1$, $k \leqslant n$, such that*

$$\sum_{j=0}^{k} (-1)^j \int_{\tau_j}^{\tau_{j+1}} u_l(t)\, dt = 0, \qquad l = 1, 2,..., n.$$

This theorem is also valid when the Lebesgue measure, dt, is replaced by any finite nonatomic measure $d\mu$ on $[0, 1]$ and $L^1[0, 1]$ is replaced by $L^1(d\mu; [0, 1])$. This form of the Hobby–Rice theorem will not concern us here.

The proof of the above result given by Hobby and Rice relies upon the antipodal mapping theorem of Borsuk. Their proof is complicated by the construction of the mapping to which the Borsuk theorem is applied. Recently, by a clever choice of the mapping function, Pinkus [15] discovered a very short proof of the Hobby–Rice theorem which avoids this difficulty. We will give below yet another proof which shows the relationship of the Hobby–Rice theorem to the Gohberg–Krein theorem on n-widths; see Lorentz [11, p. 137].

As far as we are aware, the main application of the Hobby–Rice theorem (in the generality stated above) is in its use in proving the following result of Krein [10]: There exists no finite-dimensional subspace V of $L^1[0, 1]$

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which admits for every $f \in L^1[0, 1]$ a unique best L^1 approximation by elements of V. This important result on L^1 approximation can also be proven by elementary methods, see [10].

Another use of the Hobby–Rice theorem allows one to obtain good L^1 approximations to f from $U = U(u_1, ..., u_n)$, the linear subspace spanned by $u_1(t), ..., u_n(t)$, by interpolating f at the "canonical" points $\{\tau_i : 1 \leqslant i \leqslant k\}$. This program raises several questions concerning the points whose existence is asserted in the Hobby–Rice theorem. Specifically, what is the value of k, are the points unique, can we indeed interpolate at these points, and when is the interpolant a best L^1 approximation to f?

The purpose of this paper is an attempt to give a satisfactory answer to these questions when U is a weak Chebyshev subspace (see Section 1 for a definition of this familiar notion). We will subsequently apply our results to L^1 approximation by weak Chebyshev systems and discuss their relationship to recent results of Karlin on interpolating perfect splines [7] (see also de Boor [1]). Let us emphasize here that the necessity of examining these questions for *weak* Chebyshev subspaces is dictated by the recent application of spline functions to the solution of certain extremal problems in L^∞ as discussed by Karlin [7], Micchelli and Miranker [12], and Micchelli, Rivlin, and Winograd [14]. For a discussion of the relationship of the Envelope Theorem of [12] to the result in [7, 14] see [19].

In the last section of the paper we include some remarks related to an extremal problem studied by de Boor in [2].

Now, let us give a proof of the Hobby–Rice theorem.

Proof. Clearly, we may assume without loss of generality that $u_1(t), ..., u_n(t)$ are continuous in [0, 1]. The Gohberg–Krein theorem tells us that for every $q > 1$ there exists a nontrivial polynomial p_q of degree $\leqslant n$ such that its best approximation in $L^q[0, 1]$ by the subspace $U = U(u_1, ..., u_n)$ is zero. Thus

$$\int_0^1 | p_q(t)|^{q-1} \operatorname{sgn} p_q(t)\, u_l(t)\, dt = 0, \qquad l = 1, 2, ..., n.$$

Normalize p_q so that $\int_0^1 | p_q(t)|\, dt = 1$. Since p_q has at most n zeros we may pass to the limit above, $q \to 1^+$, perhaps through a subsequence, and prove the theorem.

Let us note that the proof of the Gohberg–Krein theorem, given in Lorentz [11], uses the antipodal mapping theorem. When U is a weak Chebyshev subspace an elementary proof of the Hobby–Rice theorem is available. This proof which we present in Section 1 employs a variational argument based on a recent result of Zielke [18]; see also Zalik [17] on the existence of Chebyshev extensions.

1. Best L^1 Approximation by Weak Chebyshev Systems

Let us recall that a sequence of real-valued functions $\{u_1(x),..., u_n(x)\}$, continuous on [0, 1], is called a Chebyshev system on the open interval (0, 1) provided that the nth order determinant

$$U\begin{pmatrix} u_1 ,..., u_n \\ x_1 ,..., x_n \end{pmatrix} = \det \| u_i(x_j)\|$$

is strictly positive for $0 < x_1 < \cdots < x_n < 1$. A linearly independent sequence of continuous real-valued functions $\{u_1(x),..., u_n(x)\}$ is called a weak Chebyshev system provided that the above determinant is nonnegative for $0 < x_1 < \cdots < x_n < 1$. We will denote by U $(=U(u_1 ,..., u_n))$ the linear subspace spanned by the functions $u_1(x),..., u_n(x)$. Also, the convexity cone K $(=K(u_1 ,..., u_n))$ consists of all real-valued functions f defined on (0, 1) for which the determinant

$$U\begin{pmatrix} u_1 ,..., u_n , f \\ x_1 ,..., x_n , x_{n+1} \end{pmatrix}$$

is nonnegative for $0 < x_1 < \cdots < x_{n+1} < 1$.

We will also use the terminology that U is a weak Chebyshev subspace of $C[0, 1]$ of dimension n, provided that U is a linear space spanned by some weak Chebyshev system $\{u_1(x),..., u_n(x)\}$. When we speak of the convexity cone K $(=K(U))$ of U we mean the set $K(u_1 ,..., u_n) \cup -K(u_1 ,..., u_n)$. This set is invariant under a change of basis in U. The class of all functions in K which are continuous on the closed interval [0, 1] will be denoted by K_c $(=K_c(U))$.

Lemma 1. *Let U be a weak Chebyshev subspace of dimension n of* $C[0, 1]$. *Suppose* $h \in L^\infty[0, 1]$, $\text{meas}\{x: h(x) = 0\} = 0$, *and* $\int_0^1 h(x)\, u(x)\, dx = 0$, $u \in U$. *Then h has at least n sign changes in* [0, 1].

Proof. Suppose to the contrary that h has l sign changes with $l < n$ occurring at occurring at $0 < \tau_1 < \cdots < \tau_l < 1$; then

$$\sum_{i=0}^{n} (-1)^i \int_{\tau_i}^{\tau_{i+1}} | h(x)|\, u(x)\, dx = 0, \qquad u \in U. \tag{1}$$

For $\delta > 0$ and $U = U(u_1 ,..., u_n)$, we define

$$u_i(x; \delta) = (1/\delta(2\pi)^{1/2}) \int_0^1 \exp(-(x - t)^2/2\delta^2)\, u_i(t)\, dt, \qquad i = 1,..., n;$$

then $\{u_1(x; \delta),..., u_n(x; \delta)\}$ is a Chebyshev system and $\lim_{\delta\to 0^+} u_i(x; \delta) = u_i(x)$, $x \in (0, 1)$, $i = 1, 2,..., n$; see Karlin and Studden [9].

For any $\delta > 0$, there is a $u(x; \delta) = \sum_{j=1}^{n} a_j^{\delta} u_j(x; \delta)$ with $(-1)^i u(x; \delta) > 0$, $x \in (\tau_i, \tau_{i+1})$, $i = 0, 1,..., l$, and normalized so that $\max\{|u(x; \delta)|: 0 \leqslant x \leqslant 1\} = 1$; see Karlin and Studden [9]. Thus there exists a sequence $\delta_n \to 0^+$ such that $u(x; \delta_n) \to u(x)$, $x \in (0, 1)$, where $u(x)$ is a nontrivial element of U with $(-1)^i u(x) \geqslant 0$, $x \in (\tau_i, \tau_{i+1})$, $i = 0, 1,..., l$. We may substitute u into (1) and conclude that $\int_0^1 |h(x)| \cdot |u(x)| \, dx = 0$. This contradiction implies that $k = n$ and the lemma is proved.

Remark 1. The hypothesis that $\text{meas}\{x: h(x) = 0\} = 0$ in Lemma 1 is essential as it is possible that U may have lower dimension on subintervals of (0, 1). This frequently occurs when dealing with spline functions. However, when U is spanned by a Chebyshev system on (0, 1), that is, U is a Chebyshev subspace, this does not happen. In this case the hypothesis on h may be replaced by the weaker requirement that $\text{meas}\{x: h(x) \neq 0\} > 0$.

An immediate application of the Hobby–Rice theorem and Lemma 1 gives

LEMMA 2. *Let U be a weak Chebyshev subspace of dimension n. Then there exists a set of points, $0 = \tau_0 < \tau_1 \cdots < \tau_n < \tau_{n+1} = 1$, such that*

$$\sum_{i=0}^{n} (-1)^i \int_{\tau_i}^{\tau_{i+1}} u(x) \, dx = 0, \qquad u \in U. \tag{2}$$

We will now give a variational proof of this lemma.

Proof. Let U_δ denote the subspace spanned by the functions $\{u_1(x; \delta),..., u_n(x; \delta)\}$, and choose a function $u_{n+1}(x; \delta)$ so that $\{u_1(x; \delta),..., u_{n+1}(x; \delta)\}$ is also a Chebyshev system. The existence of a function with this property was recently proved by Zielke [18]; see also Zalik [17].

Consider the minimum problem

$$\min_{u \in U_\delta} \int_0^1 | u_{n+1}(x; \delta) - u(x)| \, dx.$$

Since, for every $u \in U_\delta$, the function $u_{n+1}(x; \delta) - u(x)$ has at most n zeros on [0, 1], we may use a standard variational argument and conclude that there exist points $0 = \tau_0^{\delta} < \tau_1^{\delta} < \cdots < \tau_k^{\delta} < \tau_{k+1}^{\delta} = 1$, $k \leqslant n$, with

$$\sum_{i=1}^{k} (-1)^i \int_{\tau_i^{\delta}}^{\tau_{i+1}^{\delta}} u_j(x; \delta) \, dx = 0, \qquad j = 1, 2,..., n. \tag{3}$$

We may easily pass to the limit in (3), perhaps through a subsequence, and conclude that there exist points $0 = \tau_0 < \tau_1 < \cdots < \tau_l < \tau_{l+1} = 1$, $l \leqslant n$, such that

$$\sum_{i=1}^{l} (-1)^i \int_{\tau_i}^{\tau_{i+1}} u(x)\, dx = 0, \qquad u \in U.$$

According to Lemma 1, $l = n$ and thus the proof is complete.

Our intention now is to give a sufficient condition on U which implies that we may interpolate at $\tau_1, \ldots, \tau_n$. To this end, we define for every $0 < x_1 < \cdots < x_n < 1$ a convex cone in R^n,

$$U[x_1, \ldots, x_n] = \{(f(x_1), \ldots, f(x_n)) : f \in K_c(U)\}.$$

The dimension of $U[x_1, \ldots, x_n]$ is defined to be the dimension of the smallest linear subspace of R^n containing $U[x_1, \ldots, x_n]$.

LEMMA 3. *Let U be a weak Chebyshev subspace of dimension n and suppose further that for every $0 < x_1 < \cdots < x_n < 1$, $U[x_1, \ldots, x_n]$ has dimension n. Then we may interpolate at the points constructed in Lemma 2; that is, if $u \in U$ and $u(\tau_i) = 0$, $i = 1, 2, \ldots, n$, then $u \equiv 0$.*

Proof. Suppose to the contrary that there exists a nontrivial element of U which vanishes at $\tau_1, \ldots, \tau_n$; then it follows that $\det \| u_i(\tau_j) \| = 0$. Hence there exist constants $c_1, \ldots, c_n$, $\sum_{j=1}^n c_j^2 \neq 0$, such that $\sum_{j=1}^n c_j u(\tau_j) = 0$, $u \in U$. Our hypothesis on the cone $U[x_1, \ldots, x_n]$ guarantees that there exists an $f \in K_c$ with $\sum_{j=1}^n c_j f(\tau_j) \neq 0$. Thus we may choose a constant d such that

$$\sum_{i=0}^{n} (-1)^i \int_{\tau_i}^{\tau_{i+1}} g(t)\, dt - d \sum_{i=1}^{n} c_i g(\tau_i) = 0, \qquad g \in U(u_1, \ldots, u_n, f). \tag{4}$$

Since $f \in K_c - U$ we may, as in the proof of Lemma 1, construct a nontrivial function $v(x) = a_0 f(x) + \sum_{i=1}^n a_i u_i(x)$ which satisfies the condition $(-1)^i v(x) \geqslant 0$, $x \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots, n$. Hence, in particular, $v(\tau_i) = 0$, $i = 1, 2, \ldots, n$, and according to (4) we obtain

$$\int_0^1 | v(x) |\, dx = 0;$$

this contradiction proves the lemma.

THEOREM 1. *Suppose $U = U(u_1, \ldots, u_n)$ is a weak Chebyshev subspace of dimension n of $C[0, 1]$ and for every $0 < x_1 < \cdots < x_n < 1$, $U[x_1, \ldots, x_n]$ has dimension n. Then every $f \in K_c(U)$ has a unique best L^1 approximation by elements of U. Furthermore, the best approximation to f is determined by the condition that it interpolates f at $\tau_1, \ldots, \tau_n$.*

Proof. The proof of this theorem is a standard consequence of Lemma 3 and the definition of K_c. The details are as follows. Let us assume for simplicity that $f \in K_c(u_1, \ldots, u_n)$. According to Lemma 3, there is a unique element $u_0 \in U$ which interpolates f at $\tau_1, \ldots, \tau_n$. We may express the difference $f(x) - u_0(x)$ as a ratio of determinants

$$f(x) - u_0(x) = \frac{\det \begin{pmatrix} u_1, \ldots, u_n, f \\ \tau_1, \ldots, \tau_n, x \end{pmatrix}}{\det \begin{pmatrix} u_1, \ldots, u_n \\ \tau_1, \ldots, \tau_n \end{pmatrix}}.$$

This equation implies that $(-1)^{i+n}(f(x) - u_0(x)) \geqslant 0$, $x \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots, n$. Hence for any $u \in U$

$$\begin{aligned} \int_0^1 |f(x) - u_0(x)| \, dx &= \sum_{i=0}^{n} (-1)^{i+n} \int_{\tau_i}^{\tau_{i+1}} (f(x) - u_0(x)) \, dx \\ &= \sum_{i=0}^{n} (-1)^{i+n} \int_{\tau_i}^{\tau_{i+1}} (f(x) - u(x)) \, dx \\ &\leqslant \int_0^1 |f(x) - u(x)| \, dx. \end{aligned}$$

Furtheremore, if for some $u \in U$ equality is achieved in the above inequality then $(-1)^{i+n}(f(x) - u(x)) \geqslant 0$, $x \in (\tau_i, \tau_{i+1})$, $i = 0, 1, \ldots, n$. Hence $f(\tau_i) = u(\tau_i)$, $i = 1, 2, \ldots, n$, and so, by Lemma 3, $u = u_0$. Thus the theorem is proved.

Let us observe that

$$\lambda(g) = \sum_{i=0}^{n} (-1)^i \int_{\tau_i}^{\tau_{i+1}} g(t) \, dt$$

is a norm one linear functional on $L^1[0, 1]$ which annihilates U, and for $f \in K_c(u_1, \ldots, u_n)$

$$\int_0^1 |f(x) - u_0(x)| \, dx = (-1)^n \lambda(f - u_0) = (-1)^n \lambda(f).$$

Thus, if $f \in K_c(u_1, \ldots, u_n) - U$, then $(-1)^n \lambda(f) > 0$.

THEOREM 2. *Let U be a weak Chebyshev subspace of dimension n of $C[0, 1]$. Suppose that the smallest closed linear subspace (relative to $L^1[0, 1]$)*

containing $K_c(U)$ contains a Chebyshev subspace of dimension n on $(0, 1)$. *Then the points of Lemma* 2 *are unique.*

Proof. Suppose to the contrary that there exists another set of points $0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1$ such that

$$\sum_{j=0}^{n} (-1)^j \int_{\xi_j}^{\xi_{j+1}} u(x)\, dx = 0, \qquad u \in U.$$

There exists an $f \in K_c$ such that

$$\sum_{j=0}^{n} (-1)^j \int_{\xi_j}^{\xi_{j+1}} f(x)\, dx \neq \sum_{j=0}^{n} (-1)^j \int_{\tau_j}^{\tau_{j+1}} f(x)\, dx,$$

since otherwise,

$$\sum_{j=0}^{n} (-1)^j \int_{\xi_j}^{\xi_{j+1}} g(t)\, dt = \sum_{j=0}^{n} (-1)^j \int_{\tau_j}^{\tau_{j+1}} g(t)\, dt$$

for all g in some Chebyshev subspace of dimension n, and, according to Lemma 1 and the remark following it, this is impossible unless $\xi_i = \tau_i$, $i = 1,..., n$. Let us assume without loss of generality that $f \in K_c(u_1, ..., u_n)$ where $U = U(u_1, ..., u_n)$. Since f is necessarily not in U the remarks following Theorem 1 tell us that

$$\sum_{j=0}^{n} (-1)^{j+n} \int_{\xi_j}^{\xi_{j+1}} f(x)\, dx > 0$$

and

$$\sum_{j=0}^{n} (-1)^{j+n} \int_{\tau_j}^{\tau_{j+1}} f(x)\, dx > 0.$$

Therefore there exists a positive constant $c \neq 1$ such that

$$\sum_{j=0}^{n} (-1)^j \int_{\tau_j}^{\tau_{j+1}} g(x)\, dx - c \sum_{j=0}^{n} (-1)^j \int_{\xi_j}^{\xi_{j+1}} g(x)\, dx = 0, \quad g \in U(u_1, ..., u_n, f). \tag{5}$$

Now, the above equation has the form $\int_0^1 h(x)\, g(x)\, dx = 0$, where h has exactly n *strict* sign changes. But $U(u_1, ..., u_n, f)$ is a weak Chebyshev subspace of dimension $n + 1$. This contradicts Lemma 1, unless we abandon our original hypothesis that the points $\tau_1, ..., \tau_n$ are not unique.

2

Let us now turn to some applications of the previous results. We denote by $S_{n,r} = S_{n,r}(x_1, ..., x_r)$ the class of spline functions of degree $n - 1$ ($n \geqslant 2$) with knots $x_1, ..., x_r$ in $(0, 1)$. Thus $S_{n,r} = U(u_1, ..., u_{n+r})$, where $u_i(t) = t^{i-1}$, $i = 1, ..., n$, $u_{n+j}(t) = (t - x_j)_+^{n-1}$, $j = 1, ..., r$, and $S_{n,r}$ is a weak Chebyshev subspace of dimension $n + r$ [9].

LEMMA 4. *The smallest linear subspace containing* $K_c(S_{n,r})$ *contains* $C^n[0, 1]$.

Proof. Every $f \in C^n[0, 1]$ is representable as

$$f(x) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j + \frac{1}{(n-1)!} \times \sum_{k=0}^{r} \int_{x_k}^{x_{k+1}} (x - t)_+^{n-1}((f^{(n)}(t))_+ - (-f^{(n)}(t))_+)\, dt,$$

$x_0 = 0$, $x_{r+1} = 1$. Thus it is sufficient to prove that the function

$$F(x) = \int_{x_k}^{x_{k+1}} (x - t)_+^{n-1} g(t)\, dt, \qquad 0 \leqslant k \leqslant n, \tag{6}$$

where $g(t) \geqslant 0$, $t \in (x_k, x_{k+1})$, is in $K_c(S_{n,r})$. The proof of this fact is easy.

Using representation (6) we compute the determinant

$$\det \begin{pmatrix} u_1, ..., u_{n+r}, F \\ t_1, ..., t_{n+r+1} \end{pmatrix}$$

to be

$$(-1)^{k+r} \int_{x_k}^{x_{k+1}} \det \begin{pmatrix} u_1, ..., u_{n+k}, u_\sigma, u_{n+k+1}, ..., u_{n+r} \\ t_1, \quad ..., \quad t_{n+r+1} \end{pmatrix} g(\sigma)\, d\sigma,$$

where $u_\sigma(t) = (t - \sigma)_+^{n-1}$. Since $\{u_1(t), ..., u_{n+k}(t), u_\sigma(t), u_{n+k+1}(t), ..., u_{n+r}(t)\}$ is a weak Chebyshev system for $\sigma \in (x_k, x_{k+1})$, we conclude that $(-1)^{k+r} F \in K_c(u_1, ..., u_{n+r})$, and the proof is finished.

The proof of Lemma 4 also shows that any $f \in C^n[0, 1]$ with $f^{(n)}$ changing signs only at $x_1, ..., x_r$ is in $K_c(S_{n,r})$.

A perfect spline function P of degree n with knots at $\xi_1, ..., \xi_r$, $0 = \xi_0 < \xi_1 < \cdots < \xi_r < \xi_{r+1} = 1$ is any function of the form

$$P(x) = \sum_{j=0}^{n-1} a_j x^j + d \sum_{j=0}^{r} (-1)^j \int_{\xi_j}^{\xi_{j+1}} (x - t)_+^{n-1}\, dt.$$

Note that $P \in C^{n-1}(-\infty, \infty)$ and $(-1)^j P^{(n)}(x) = d(n-1)!$ for $x \in (\xi_j, \xi_{j+1})$, $j = 0, 1,..., r$.

THEOREM 3. *There exists a unique perfect spline P_r with $n + r$ knots $0 = \tau_0 < \tau_1 < \cdots < \tau_{n+r} < \tau_{n+r+1} = 1$ such that $P_r^{(i)}(0) = P_r^{(i)}(1) = 0$, $i = 0, 1,..., n-1$, $P_r(x_j) = 0$, $j = 1, 2,..., r$, normalized so that $P_r^{(n)}(0) = 1$. Furthermore, whenever f is a continuous function in the convexity cone of $S_{n,r}(x_1,..., x_r)$, f has a unique best L^1 approximation from $S_{n,r}$, and it is determined by interpolating f at the knots of $P_r(x)$.*

Proof. Lemma 4 clearly indicates that the hypothesis of Theorem 2 is satisfied for $S_{n,r}$. Thus according to Theorems 1 and 2 the unique best L^1 approximation to f interpolates f at the unique points $\tau_1,..., \tau_{n+r}$ determined by the condition

$$\sum_{i=0}^{n+r} (-1)^i \int_{\tau_i}^{\tau_{i+1}} s(x)\, dx = 0, \qquad s \in S_{n,r}. \tag{7}$$

The proof is completed by observing that the function

$$P_r(x) = (1/(n-1)!) \sum_{i=0}^{n+r} (-1)^i \int_{\tau_i}^{\tau_{i+1}} (x-t)_+^{n-1}\, dt$$

satisfies the conditions of the corollary.

When $r = 0$, the unique points which satisfy (7) are the interior extrema of Chebyshev polynomial of degree $n + 1$; see Rivlin [16]. Thus

$$P_0(x) = (1/(n-1)!) \int_0^1 \operatorname{sgn} T_{n+1}(2t-1)(x-t)_+^{n-1}\, dt,$$

an observation due to Louboutin [5].

The existence and uniqueness of P_r was first proved by Karlin in [7].

Recently, a number of papers have appeared which treat the question of uniqueness of L^1 approximation [3, 4] by spline functions with fixed knots. These papers show that any continuous function has a unique L^1 approximation by spline functions with fixed knots. Our theorem gives a characterization of the best approximation when f is a continuous function in the convexity cone of $S_{n,r}$.

Karlin also proves in [7] the following uniqueness theorem which we will also show to be a consequence of Theorem 2.

Before we state Karlin's result we record below a lemma which we will have several occasions to use.

We will say a vector $y = (y_1,..., y_{n+1})$ in R^{n+1} weakly (strictly) alternates provided that y is nonzero and $y_i y_{i+1} \leq 0$ ($y_i y_{i+1} < 0$), $i = 1, 2,..., n$. In addition, we define $\operatorname{supp}(y) = \{j: y_j \neq 0\}$.

LEMMA 5. *Let y be any $n+1$ vector which (weakly) strictly alternates and suppose $u_1(x),..., u_{n+1}(x)$ are linearly independent continuous functions on* [0, 1]. *Then*

$$U(y) = \left\{\sum_{j=1}^{n+1} a_j u_j(x): \sum_{j=1}^{n+1} a_j y_j = 0\right\}$$

is a (weak) Chebyshev subspace of dimension n provided that $\{u_1(x),..., u_{j-1}(x), u_{j+1}(x),..., u_{n+1}(x)\}$ is a (weak) Chebyshev system for any $j \in \operatorname{supp}(y)$. Furthermore, any $f \in \bigcap_j \{K_c(u_1,..., u_{j-1}, u_{j+1},..., u_{n+1}): j \in \operatorname{supp}(y)\}$ is in the convexity cone of $U(y)$.

COROLLARY 1 (Karlin). *Given any data $y_1,..., y_{n+r+1}$ and points $0 = x_1 < x_2 < \cdots < x_{n-r+1} = 1$. If the divided difference of $y_i,..., y_{i+n}$ at $x_i,..., x_{i+n}$ weakly alternates and $r \leqslant n$, then there exists a unique perfect spline P with r knots such that $P(x_i) = y_i$, $i = 1, 2,..., n+r+1$.*

Proof. We denote the divided diffrerence of $y_i,..., y_{i+n}$ at $x_i,..., x_{i+n}$ by z_i. If $P(x_i) = y_i$, $i = 1, 2,..., n+r+1$ and

$$P(x) = \sum_{j=0}^{n-1} a_j x^j + d \sum_{j=0}^{r} (-1)^j \int_{\xi_j}^{\xi_{j+1}} (x-t)_+^{n-1}\, dt,$$

then

$$z_i = d \sum_{j=0}^{n} (-1)^j \int_{\xi_j}^{\xi_{j+1}} M(x_i,..., x_{i+n}, t)\, dt,$$

where $M(x_i,..., x_{i+n}, t)$ is a B-spline, defined to be the divided difference of $(x-t)_+^{n-1}$ at $x = x_i,..., x_{i+n}$. Hence the corollary will follow from Lemma 2 and Theorem 2 provided that

$$V = \left\{\sum_{j=1}^{r+1} a_j M(x_j,..., x_{j+n}, t): \sum_{j=1}^{r+1} a_j z_j = 0\right\}$$

is a weak Chebyshev subspace of dimension r and its associated convexity cone contains a Chebyshev system of dimension r. The proof of these facts begins with the observation that every subsequence of B-splines span a weak Chebyshev subspace [8]. Thus, since the vector $z = (z_1,..., z_{r+1})$ weakly alternates, we conclude from Lemma 5 that V is a weak Chebyshev subspace.

We will now show that $S_{n,n+r-1}(x_2,..., x_{n+r}) \subseteq K_c(V)$. Then, since $n \geqslant r$ our requirement that $K_c(V)$ contain a Chebyshev subspace of dimension r will certainly be satisfied.

Choose any a and b, $a < 0 < 1 < b$, and extend the sequence $x_1,..., x_{n+r+1}$ so that a and b occur with multiplicity $n-1$. We label the resulting partition by $x_{-n+1} \leqslant \cdots \leqslant x_{2n+r}$ and observe that as a consequence of Lemma 5 $M(x_i, ..., x_{i+n}, t) \in K_c(V)$, $i = -n+1, ..., n+r$. The subspace spanned by these functions restricted to $[0, 1]$ is $S_{n,n+r-1}(x_2, ..., x_{n+r})$. Thus the proof is complete.

Following a route similar to that taken in proof of Theorem 3 we obtain another consequence of Theorems 1 and 2.

We will call a continuous real-valued kernel $K(x, y)$ a nondegenerate totally positive kernel of order $r + 1$ provided that

$$K\binom{x_1, ..., x_l}{y_1, ..., y_l} = \det \| K(x_i, y_j) \| \geqslant 0$$

for any $0 < x_1 < \cdots < x_l < 1$, $0 < y_1 < \cdots < y_l < 1$, $l = 1, ..., r+1$, and, in addition, $\dim U(K(x_1, \cdot), ..., K(x_r, \cdot)) = \dim U(K(\cdot, x_1), ..., K(\cdot, x_r)) = r$, for any $0 < x_1 < \cdots < x_r < 1$.

THEOREM 4. *Let $K(x, y)$ be a nondegenerate totally positive kernel of order $r + 1$. Then given any $x_1, ..., x_r$, $0 < x_1 < \cdots < x_r < 1$, there exist points $\tau_1, ..., \tau_r$, $0 < \tau_1 < \cdots < \tau_r < 1$, such that every f in the convexity cone of $U = U(K(\cdot, x_1), ..., K(\cdot, x_r))$ has a unique best L^1 approximation from U which is determined by interpolating f at $\tau_1, ..., \tau_r$. Furthermore, if the set of functions $\{K(x, t) : t \in [0, 1]\}$ is dense in $L^1[0, 1]$ then the points $\tau_1, ..., \tau_r$ are unique.*

The proof of this theorem depends on the following observation. Let $d\mu = \sum_{j=0}^{r+1} (-1)^j \, d\mu_j$, where $d\mu_j$ is a finite positive measure supported on $[x_j, x_{j+1}]$ $(x_0 = 0,\ x_{r+1} = 1)$, $j = 0, 1, ..., r$. Then $f(x) = \int_0^1 K(x, t) \, d\mu(t)$ is in the convexity cone of $U(K(\cdot, x_1), ..., K(\cdot, x_r))$.

3. An Extremal Problem

In this section we require the following lemma.

LEMMA 6. *Let $y = (y_1, ..., y_{n+1})$ be a nonzero vector in R^{n+1} and let $\{u_1(x), ..., u_{n+1}(x)\}$ be a weak Chebyshev system on $[0, 1]$. Then there exist points $\tau_1, ..., \tau_k$, $0 = \tau_0 < \tau_1 < \cdots < \tau_k < \tau_{k+1} = 1$, $k \leqslant n$, and a constant $\lambda = \lambda(y) \neq 0$ such that*

$$\sum_{j=0}^{k} (-1)^j \int_{\tau_j}^{\tau_{j+1}} u_i(t) \, dt = \lambda y_i, \qquad i = 1, 2, ..., n+1.$$

Furthermore,

$$|\lambda| = \min_{\sum_{i=1}^{n+1} a_i y_i = 1} \int_0^1 \left| \sum_{i=1}^{n+1} a_i u_i(t) \right| dt,$$

and

$$|\lambda|^{-1} = \min_{\int_0^1 h(t) u_i(t)\,dt = y_i,\, i=1,2,\ldots,n+1} \|h\|_\infty ,$$

where $\|h\|_\infty = \text{ess sup}\{|h(x)|: 0 \leqslant x \leqslant 1\}$.

The first part of this lemma follows from Lemma 1 and the Hobby–Rice theorem. When $\{u_1(x),\ldots, u_{n+1}(x)\}$ is a Chebyshev system this result is due to Krein and it is an essential ingredient in his analysis of the L-problem [10]. For a weak Chebyshev system, we may merely "apply some heat" as in Lemma 2 to provide a variational proof of Lemma 6. The essence of this observation is contained in [1]. Another application of Krein's L-problem for weak Chebyshev systems is discussed in [12]. Finally, we remark that an important point in the proof of Lemma 6 is the fact that

$$|\lambda_\delta| = \min_{\sum_{j=1}^{n+1} a_j y_j = 1} \int_0^1 \left| \sum_{j=1}^{n+1} a_j u_j(t_j\,;\delta) \right| dt,$$

where $u_1(t;\delta),\ldots, u_{n+1}(t;\delta)$ are defined in Lemma 2, converges to $|\lambda|$ as $\delta \to 0^+$. Using this fact we prove

THEOREM 5. *Suppose that the sequence* $\{u_1(x),\ldots, u_{j-1}(x), u_{j+1}(x),\ldots, u_{n+1}(x)\}$ *forms a weak Chebyshev system on* $(0, 1)$ *for all* $j = 0, 1,\ldots, n+1$; *then*

$$|\lambda(y)| \geqslant |\lambda(e)|, \qquad e = (1, -1,\ldots, (-1)^n), \tag{8}$$

provided $\|y\|_\infty = \max_{1\leqslant i\leqslant n+1} |y_i| \leqslant 1$.

If the sequence $\{u_1(x),\ldots, u_{j-1}(x), u_{j+1}(x),\ldots, u_{n+1}(x)\}$ *is a Chebyshev system for all* $j = 0, 1,\ldots, n+1$, *then equality is achieved above if and only if* $y = \pm e$.

Proof. According to our remarks preceding Theorem 5 may assume without loss of generality that $\{u_1(x),\ldots, u_{j-1}(x), u_{j+1}(x),\ldots, u_{n+1}(x)\}$ is a Chebyshev system for $j = 0, 1,\ldots, n+1$. We wish to prove that if $\|y\|_\infty \leqslant 1$ and $d = |\lambda(y)|/|\lambda(e)| \leqslant 1$, then $y = \pm e$. Assume to the contrary that $d \leqslant 1$, $\|y\|_\infty \leqslant 1$, and $y \neq \pm e$. Define $h_y(t) = \lambda^{-1}(y)(-1)^j$, $\tau_j < t < \tau_{j+1}$, $j = 0, 1,\ldots, k$, $k \leqslant n$, where $\tau_1,\ldots, \tau_k$ are defined in Lemma 5. We choose a sign σ, $\sigma^2 = 1$, so that $h = \sigma\, dh_y - h_e$ vanishes in a neighborhood of zero. Clearly, h has fewer than n sign changes and is not identically zero. However,

$$\int_0^1 h(t)\, u_j(t)\, dt = z_j\,, \qquad j = 1, 2,\ldots, n+1,$$

where $z = \sigma\, dy - e$ is a weakly alternating vector. But Lemma 5 and Remark 1 imply that h has at least n sign changes. This contradiction proves that $y = \pm e$ and the theorem.

Observe that, according to Lemmas 1, 5, and 6, there exist points $0 = \eta_0 < \eta_1 < \cdots < \eta_n < \eta_{n+1} = 1$ such that

$$\sum_{j=0}^{n} (-1)^j \int_{\eta_j}^{\eta_{j+1}} u_i(x)\,dx = \lambda(e)(-1)^i, \qquad i = 1, 2,..., n+1. \tag{9}$$

Let us now consider some applications of Theorem 5.

Let

$$W_\infty^n[0, 1] = \{f: f^{(n-1)} \text{ absolutely continuous}, f^{(n)} \in L^\infty[0, 1]\},$$

and $0 \leqslant x_1 < x_2 < \cdots < x_{n+r+1} \leqslant 1$. Define

$$C_n(y) = \{f: f \in W_\infty^n[0, 1], f(x_i) = y_i\,, i = 1, 2,..., n+r+1\};$$

then we have

COROLLARY 2. *Let* $z_i = [y_i\,,..., y_{i+n}]$ *be the divided difference of* $y_i\,,..., y_{i+n}$ *at* $x_i\,,..., x_{i+n}\,, i = 1, 2,..., r+1$; *then*

$$\max_{\|z\|_\infty \leqslant 1} \min_{f \in C_n(y)} \| f^{(n)} \|_\infty = \| Q^{(n)} \|_\infty\,,$$

where Q *is a perfect spline with* r *knots in* $(0, 1)$ *such that*

$$[Q(x_i),..., Q(x_{i+n})] = (-1)^i, \qquad i = 1, 2,..., r+1.$$

In [2], de Boor gives upper and lower bounds for $\| Q^{(n)} \|_\infty$ which are independent of $x_1\,,..., x_{n+r+1}$.

Let $K(x, y)$ be a nondegenerate totally positive kernel of order $r+1$. In [13], it was shown that there exist $0 = \eta_0{}^* < \eta_1{}^* < \cdots < \eta_r{}^* < \eta_{r+1}^* = 1$, $0 \leqslant x_1{}^* < \cdots < x_{r+1}^* \leqslant 1$ such that the function

$$G(x) = \sum_{j=0}^{r} (-1)^j \int_{\eta_j{}^*}^{\eta_{j+1}^*} K(x, t)\,dt$$

equioscillates $r+1$ times on $x_1{}^*,..., x_{r+1}^*$; that is,

$$G(x_i{}^*) = (-1)^{i+1} \| G \|_\infty\,, \qquad i = 1, 2,..., r+1.$$

Furthermore, $d_r(\mathscr{K}) = \| G \|_\infty$, where $d_r(\mathscr{K})$ is the rth width of the set

$$\mathscr{K} = \left\{ \int_0^1 K(x, t)\,h(t)\,dt \colon \| h \|_\infty \leqslant 1 \right\}$$

in $L^\infty[0, 1]$. Also, for any $f \in \mathscr{K}$, and $0 < x_1 < \cdots < x_r < 1$ for which the vector $(f(x_1),..., f(x_{r+1}))$ strictly alternates we have

$$\min_{1 \leqslant i \leqslant r+1} | f(x_i) | \leqslant \| G \|_\infty\,.$$

We may conclude from these remarks and Theorem 5 that

$$\min_{0\leqslant x_1<\cdots<x_{r+1}\leqslant 1} \max_{\|y\|_\infty\leqslant 1} \min_{\int_0^1 K(x_i,t)h(t)\,dt=y_i,\, i=1,2,\ldots,r+1} \|h\|_\infty = 1/d_r(\mathscr{K}).$$

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References

1. C. de Boor, A remark concerning perfect splines, *Bull. Amer. Math. Soc.* **80** (1974), 724–727.
2. C. de Boor, How small can one make the derivative of an interpolating function, MRC Tech. Summ. Report #1425, U. S. Army, Univ. of Wisc., 1974.
3. M. P. Carroll and D. Braess, On uniqueness of L^1 approximation for certain families of spline functions, *J. Approximation Theory* **12** (1974), 362–364.
4. R. V. Galkin, The uniqueness of the element of best mean approximation to a continuous function using splines with fixed nodes, *Math. Notes* **15** (1974), 3–8.
5. G. Glaeser, Ed., Le prolongateur de Whitney, 1967, Université de Rennes.
6. C. R. Hobby and J. R. Rice, A moment problem in L_1 approximation, *Proc. Amer. Math. Soc.* **65** (1965), 665–670.
7. S. Karlin, Interpolation properties of generalized perfect splines and the solution of certain extremal problems I, *Trans. Amer. Math. Soc.* **206** (1975), 25–66.
8. S. Karlin, "Total Positivity," Vol. 1, Stanford Univ. Press, Stanford, 1968.
9. S. Karlin and W. J. Studden, "Tchebycheff Systems: with Applications In Analysis and Statistics," Interscience, New York, 1966.
10. M. Krein, The L-problem in abstract linear normed space, *in* "Some Questions in the Theory of Moments" (N. I. Akiezer, M. Krein, Eds.), Translations of Mathematical Monographs, Vol. 2, Amer. Math. Soc., Providence, R. I., 1962.
11. G. G. Lorentz, "Approximation of Functions," Holt, Rinehart, and Winston, New York, 1966.
12. C. A. Micchelli and W. L. Miranker, High order search methods for finding roots, *J. Assoc. Comput. Mach.* **22** (1975), 51–60.
13. C. A. Micchelli and A. Pinkus, On n-widths in L^∞, *Trans. Amer. Math. Soc.*, to appear.
14. C. A. Micchelli, T. J. Rivlin, and S. Winograd, Optimal recovery of smooth functions, *Numer. Math.* **26** (1976), 191–200.
15. Allan Pinkus, A simple proof of the Hobby–Rice theorem, *Proc. Amer. Math. Soc.*, to appear.
16. T. J. Rivlin, "An Introduction to Approximation," Blaisdell, Waltham, Mass., 1969.
17. R. A. Zalik, Existence of Tchebycheff extensions, *J. Math. Anal. Appl.* **51** (1975), 68–75.
18. R. Zielke, Alternation properties of Tchebycheff-systems and the existence of adjoined functions, *J. Approximation Theory* **10** (1974), 172–184.
19. C. A. Micchelli and T. J. Rivlin, A survey of optimal recovery, *in* "Proceedings of the International Conference of Optimal Estimation in Approximation Theory," to appear.